

THE η -EINSTEIN CONDITION ON INDEFINITE \mathcal{S} -MANIFOLDS

LETIZIA BRUNETTI

ABSTRACT. An η -Einstein condition is introduced in the context of indefinite $g.f.f$ -manifolds, and a few Schur-type lemmas for indefinite \mathcal{S} -manifolds are provided.

1. INTRODUCTION AND PRELIMINARIES

The notion of f -structure on a $(2n + s)$ -dimensional manifold M , i.e. a $(1, 1)$ -type tensor field φ on M of constant rank $2n$ such that $\varphi^3 + \varphi = 0$, was firstly introduced in 1963 by K. Yano ([19]) as a generalization of both (almost) contact (for $s = 1$) and (almost) complex structures (for $s = 0$). During the subsequent years, this notion has been furtherly developed by several authors ([1], [2], [10], [11], [12], [15], [16]). Among them, H. Nakagawa in [15] and [16] introduced the notion of framed f -manifold, later developed and studied by S.I. Goldberg and K. Yano ([10], [11]) and others with the denomination of globally framed f -manifolds.

A manifold M is said to be a *globally framed f -manifold* (briefly *$g.f.f$ -manifold*) if it carries a globally framed f -structure, that is an f -structure φ such that the subbundle $\ker(\varphi)$ is parallelizable. If $\text{rank}(\ker(\varphi)) = s \geq 1$ the existence of a $g.f.f$ -structure on M is equivalent to the existence of s linearly independent global vector fields ξ_α and 1-forms η^α , $\alpha \in \{1, \dots, s\}$, such that

$$\varphi^2 = -I + \eta^\alpha \otimes \xi_\alpha \quad \text{and} \quad \eta^\alpha(\xi_\beta) = \delta_\beta^\alpha, \quad (1.1)$$

where I is the identity mapping. We point out that this kind of structure is also known as “ f -structure with complemented frames” ([1], [5]), or “almost r -contact structure” ([18]).

From (1.1) it follows that $\varphi\xi_\alpha = 0$ and $\eta^\alpha \circ \varphi = 0$, for any $\alpha \in \{1, \dots, s\}$. Moreover, $TM = \text{Im}(\varphi) \oplus \text{span}(\xi_1, \dots, \xi_s)$, where $\text{Im}(\varphi)$ is a distribution on M of even rank $r = 2n$ on which φ acts as an almost complex tensor field, so

1991 *Mathematics Subject Classification*. Primary 53C15, 53C50; Secondary 53C25, 53B30.

Key words and phrases. Indefinite \mathcal{S} -manifold. η -Einstein condition. Schur lemma. Semi-Riemannian manifold.

that $\dim(M) = 2n + s$. Each ξ_α is said to be a *characteristic vector field* of the structure. A *g.f.f*-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha)$ is called *normal* if the $(1, 2)$ -type tensor field $N = [\varphi, \varphi] + 2d\eta^\alpha \otimes \xi_\alpha$ vanishes identically ([12]).

Globally framed *f*-structures can always be considered together with an associated Riemannian metric ([1], [19]), while for general indefinite metrics some restrictions on the signature have to be observed. Such restrictions disappear in the case of *g.f.f*-manifolds endowed with Lorentzian metrics (see p. 214 of [6]). More recently, a study of a particular class of *g.f.f*-manifolds endowed with an indefinite metric has been carried out in ([3]). Following [6, 8, 3], we say that an indefinite metric g on a *g.f.f*-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha)$ is *compatible* with the *g.f.f*-structure $(\varphi, \xi_\alpha, \eta^\alpha)$ if

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^s \varepsilon_\alpha \eta^\alpha(X) \eta^\alpha(Y), \quad (1.2)$$

for all $X, Y \in \Gamma(TM)$, where $\varepsilon_\alpha = g(\xi_\alpha, \xi_\alpha) = \pm 1$. Then, the manifold M is said to be an *indefinite (metric) g.f.f-manifold* with structure $(\varphi, \xi_\alpha, \eta^\alpha, g)$. From (1.2) we easily get

$$g(X, \xi_\alpha) = \varepsilon_\alpha \eta^\alpha(X) \quad \text{and} \quad g(X, \varphi Y) = -g(\varphi X, Y), \quad (1.3)$$

for any $X, Y \in \Gamma(TM)$ and any $\alpha \in \{1, \dots, s\}$. Furthermore, $\text{Im}(\varphi)$ is orthogonal to $\text{span}(\xi_1, \dots, \xi_s)$, and since $g(\varphi X, \varphi Y) = g(X, Y)$, for any $X, Y \in \text{Im}(\varphi)$, then the signature of g on $\text{Im}(\varphi)$ is $(2p, 2q)$, with $2p + 2q = 2n$. In [6] it is proved that there always exists a Lorentzian metric g associated with a *g.f.f*-manifold, and in this case exactly one of the characteristic vector fields has to be unit timelike and the restriction of g to $\text{Im}(\varphi)$ has Riemannian signature.

The 2-form Φ on M defined by $\Phi(X, Y) = g(X, \varphi Y)$ is called the *fundamental 2-form* of the indefinite *g.f.f*-manifold. If $\Phi = d\eta^\alpha$, for any $\alpha \in \{1, \dots, s\}$, the manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ is said to be an *indefinite almost \mathcal{S} -manifold*. Finally, a normal indefinite almost \mathcal{S} -manifold is, by definition, an *indefinite \mathcal{S} -manifold*. As proved in [3], in an indefinite \mathcal{S} -manifold the covariant derivative of φ satisfies the identity

$$(\nabla_X \varphi)Y = g(\varphi X, \varphi Y) \bar{\xi} + \bar{\eta}(Y) \varphi^2 X, \quad (1.4)$$

where $\bar{\xi} = \sum_{\alpha=1}^s \xi_\alpha$ and $\bar{\eta} = \sum_{\alpha=1}^s \varepsilon_\alpha \eta^\alpha$, from which it easily follows that, for any $\alpha, \beta \in \{1, \dots, s\}$,

$$\nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi X \quad \text{and} \quad \nabla_{\xi_\alpha} \xi_\beta = 0, \quad (1.5)$$

as well as that each ξ_α is a Killing vector field. In particular, for $s = 1$ one finds again the notion of indefinite Sasakian manifold ([17]).

In the Riemannian setting, the notion of \mathcal{S} -manifold, together with other remarkable classes of *g.f.f*-manifolds, appears in [1]. It has been developed by several authors and for further properties we refer the reader to [1], [2], [5] and

[7], where the notion of almost \mathcal{S} -manifold is introduced. The generalization of this notion to the semi-Riemannian setting is given in [3].

The main purpose of this short note is to extend the notion of η -Einstein $g.f.f$ -structure to the semi-Riemannian setting, by a suitable generalization of the definition given in [14]. We provide it in Section 2, pointing out the main differences between our definition and that contained in [14], and give a first Schur-type lemma. Based on it, in Section 3, we prove a second Schur-type result for indefinite \mathcal{S} -manifolds with pointwise constant φ -sectional curvature and its suitable consequence. In a forthcoming paper, we are going to develop and apply the results contained here to the study of the φ -null Osserman condition on Lorentzian \mathcal{S} -manifolds.

In what follows, all manifolds, tensor fields and maps are assumed to be smooth. Moreover, all manifolds are supposed to be connected and, according to [13], for the curvature tensors of a semi-Riemannian manifold (M, g) we put

$$R(X, Y, Z, W) = g(R(Z, W)Y, X) = g([\nabla_Z, \nabla_W] - \nabla_{[Z, W]})Y, X),$$

for any vector fields X, Y, Z, W on M . Finally, for any $p \in M$ and any linearly independent vectors $x, y \in T_p M$ spanning a non-degenerate plane $\pi = \text{span}(x, y)$, that is $\Delta(\pi) = g_p(x, x)g_p(y, y) - g_p(x, y)^2 \neq 0$, the sectional curvature of (M, g) at p with respect to π is, by definition, the real number

$$k_p(\pi) = k_p(x, y) = \frac{R_p(x, y, x, y)}{\Delta(\pi)}.$$

2. THE η -EINSTEIN CONDITION FOR INDEFINITE $g.f.f$ -MANIFOLDS.

Let us now state some preliminary properties of the curvature tensor field of an indefinite \mathcal{S} -manifold.

Proposition 2.1. *Let $(M, \varphi, \xi^\alpha, \eta_\alpha, g)$, $1 \leq \alpha \leq s$, be a $(2n + s)$ -dimensional indefinite \mathcal{S} -manifold. The following identities hold, for any $X, Y, Z \in \Gamma(TM)$, any $U, V \in \text{span}(\xi_1, \dots, \xi_s)$ and any $\alpha, \beta, \gamma \in \{1, \dots, s\}$.*

- (1) $R(X, Y, \xi_\alpha, Z) = \varepsilon_\alpha \{ \bar{\eta}(X)g(\varphi Y, \varphi Z) - \bar{\eta}(Y)g(\varphi X, \varphi Z) \};$
- (2) $R(\xi_\beta, Y, \xi_\alpha, Z) = \varepsilon_\beta \varepsilon_\alpha g(\varphi Y, \varphi Z);$
- (3) $R(\xi_\beta, \xi_\gamma, \xi_\alpha, Z) = 0;$
- (4) $R(\varphi X, \varphi Y, \xi_\alpha, Z) = 0;$
- (5) $R(U, Y, V, Z) = \bar{\eta}(U)\bar{\eta}(V)g(\varphi Y, \varphi Z).$

where, $\varepsilon_\alpha = g(\xi_\alpha, \xi_\alpha) = \pm 1$ for any $\alpha \in \{1, \dots, s\}$, and $\bar{\eta} = \sum_{\alpha=1}^s \varepsilon_\alpha \eta^\alpha$.

Proof. With straightforward calculations, using (1.4), one gets (1). The identities (2), (3) and (4) are easy consequences of (1), while (5) follows from (2). \square

As a consequence of the above properties, computing the Ricci tensor field $Ric(X, \xi_\alpha)$, for any $X \in \Gamma(TM)$ and any $\alpha \in \{1, \dots, s\}$, we get

$$\begin{aligned} Ric(X, \xi_\alpha) &= \sum_{i=1}^n \varepsilon_i \{R(X, E_i, \xi_\alpha, E_i) + R(X, \varphi E_i, \xi_\alpha, \varphi E_i)\} \\ &\quad + \sum_{\beta=1}^s \varepsilon_\beta R(X, \xi_\beta, \xi_\alpha, \xi_\beta) \\ &= \sum_{i=1}^n \varepsilon_i \varepsilon_\alpha \bar{\eta}(X) \{g(\varphi E_i, \varphi E_i) + g(E_i, E_i)\} = 2n \varepsilon_\alpha \bar{\eta}(X) \end{aligned} \quad (2.1)$$

where $(E_i, \varphi E_i, \xi_\beta)$, $i \in \{1, \dots, n\}$ and $\beta \in \{1, \dots, s\}$, is any local orthonormal φ -adapted frame. Hence, using argumentations similar to those in [14], we see at once that indefinite \mathcal{S} -manifolds can not be Einstein. Therefore, we introduce the notion of η -Einstein condition on an indefinite $g.f.f$ -manifold as follows.

Definition 2.2. An indefinite $g.f.f$ -manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ is said to be η -Einstein if there exist two functions $h, k \in \mathfrak{F}(M)$ such that

$$Ric(X, Y) = hg(\varphi X, \varphi Y) + k\bar{\eta}(X)\bar{\eta}(Y), \quad (2.2)$$

for any $X, Y \in \Gamma(TM)$, where $\bar{\eta} = \sum_{\alpha=1}^s \varepsilon_\alpha \eta^\alpha$.

Remark 2.3. Using (2.1), from (2.2) one deduces that a $(2n + s)$ -dimensional indefinite \mathcal{S} -manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ is η -Einstein if and only if (2.2) holds with $k = 2n$, that is

$$Ric(X, Y) = hg(\varphi X, \varphi Y) + 2n\bar{\eta}(X)\bar{\eta}(Y). \quad (2.3)$$

for any $X, Y \in \Gamma(TM)$.

Remark 2.4. It is easy to see that our definition reduces to the one given in [14], when the signature of the metric is Euclidean. Indeed, in this case, we have $\varepsilon_\alpha = 1$, for any $\alpha \in \{1, \dots, s\}$, and (2.3) perfectly agrees with the condition (1.12) of [14], up to a multiplying factor.

Nevertheless, it is worth noting that (2.3) cannot be obtained from (1.12) of [14] simply by inserting the ε_α 's. Indeed, referring to [14] where the authors denote by \tilde{S} the Ricci tensor field, if we replace each $\tilde{\eta}_x$ by $\varepsilon_x \tilde{\eta}_x$ in (1.12), then we will get

$$\begin{aligned} \tilde{S}(\tilde{X}, \tilde{Y}) &= a(\tilde{G}(\tilde{X}, \tilde{Y}) + \sum_{x \neq y} \varepsilon_x \tilde{\eta}_x(\tilde{X}) \varepsilon_y \tilde{\eta}_y(\tilde{X})) \\ &\quad + b(\sum_x \tilde{\eta}_x(\tilde{X}) \tilde{\eta}_x(\tilde{Y}) + \sum_{x \neq y} \varepsilon_x \tilde{\eta}_x(\tilde{X}) \varepsilon_y \tilde{\eta}_y(\tilde{X})), \end{aligned}$$

with $a + b = 2n$ (up to a multiplying factor). The above expression is not equivalent to (2.3), due to the term $\sum_x \tilde{\eta}_x(\tilde{X})\tilde{\eta}_x(\tilde{Y})$, which does not agree with the analogous term obtained from (2.3) by expanding it with the use of (1.2).

Remark 2.5. Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an η -Einstein indefinite \mathcal{S} -manifold. Then the scalar curvature τ is given by

$$\begin{aligned} \tau &= \sum_{i=1}^n \varepsilon_i \{ \text{Ric}(E_i, E_i) + \text{Ric}(\varphi E_i, \varphi E_i) \} + \sum_{\beta=1}^s \varepsilon_\beta \text{Ric}(\xi_\beta, \xi_\beta) \\ &= 2nh + 2n \sum_{\beta=1}^s \varepsilon_\beta = 2n(h + \varepsilon), \end{aligned}$$

where $(E_i, \varphi E_i, \xi_\beta)$, $i \in \{1, \dots, n\}$ and $\beta \in \{1, \dots, s\}$, is any local orthonormal φ -adapted frame and $\varepsilon = \sum_{\beta=1}^s \varepsilon_\beta$.

Now we state the first Schur-type lemma for an η -Einstein indefinite \mathcal{S} -manifold.

Theorem 2.6. Let $(M^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$, $n \geq 2$ and $s \geq 1$, be an η -Einstein indefinite \mathcal{S} -manifold. Then the function h in (2.3) is constant.

Proof. Given $p \in M$, let $(E_i)_{i \in \{1, \dots, 2n+s\}}$ be a local orthonormal frame on a neighborhood \mathfrak{U} of p such that $(\nabla_{E_i} E_j)_p = 0$, for any $i, j \in \{1, \dots, 2n+s\}$. Then, the Second Bianchi Identity, evaluated at the point p , has the form $\sigma_{(m,i,j)} E_m(R(E_i, E_j, E_k, E_l)) = 0$, for any $m, i, j, k \in \{1, \dots, 2n+s\}$. Putting $i = k$ and $j = l$, multiplying by $\varepsilon_i = g(E_i, E_i)$ and taking the sum over all $i \in \{1, \dots, 2n+s\}$, we get $E_m(\text{Ric}(E_j, E_j) - 2E_j(\text{Ric}(E_j, E_m))) = 0$, for any $m, j \in \{1, \dots, 2n+s\}$. Multiplying again by ε_j and taking the sum over all $j \in \{1, \dots, 2n+s\}$, by Remark 2.5, we obtain

$$2nE_m(h) - 2 \sum_{j=1}^{2n+s} \varepsilon_j E_j(\text{Ric}(E_j, E_m)) = 0. \quad (2.4)$$

On the other hand, by (2.3), one has

$$\begin{aligned} E_j(\text{Ric}(E_j, E_m)) &= E_j(h)g(\varphi E_j, \varphi E_m) + hE_j(g(\varphi E_j, \varphi E_m)) \\ &\quad + 2nE_j(\bar{\eta}(E_j)\bar{\eta}(E_m)). \end{aligned} \quad (2.5)$$

Let us now calculate each term of the above identity separately. About the first one, using (1.2), we get, for any $m \in \{1, \dots, 2n+s\}$,

$$\sum_{j=1}^{2n+s} \varepsilon_j E_j(h)g(\varphi E_j, \varphi E_m) = E_m(h) - \sum_{\alpha=1}^s \eta^\alpha(E_m)\xi_\alpha(h). \quad (2.6)$$

About the second term, by (1.3) and (1.5), we have, at the point p , $E_j(\eta^\alpha(E_j)) = -g(E_j, \varphi E_j) = 0$. Using (1.2) again, we get, for any $m \in \{1, \dots, 2n+s\}$,

$$\begin{aligned} \sum_{j=1}^{2n+s} \varepsilon_j E_j(g(\varphi E_j, \varphi E_m)) &= - \sum_{j=1}^{2n+s} \sum_{\alpha=1}^s \varepsilon_j \varepsilon_\alpha E_j(\eta^\alpha(E_j) \eta^\alpha(E_m)) \\ &= - \sum_{j=1}^{2n+s} \sum_{\alpha=1}^s \varepsilon_j g(E_j, \xi_\alpha) g(\varphi E_m, E_j) \\ &= -g(\bar{\xi}, \varphi E_m) = 0. \end{aligned} \tag{2.7}$$

About the third term, since $E_j(\bar{\eta}(E_j)) = 0$, we have, for any $m \in \{1, \dots, 2n+s\}$,

$$\begin{aligned} \sum_{j=1}^{2n+s} \varepsilon_j E_j(\bar{\eta}(E_j) \bar{\eta}(E_m)) &= \sum_{j=1}^{2n+s} \sum_{\alpha=1}^s \varepsilon_\alpha \varepsilon_j g(E_j, \bar{\xi}) g(\varphi E_m, E_j) \\ &= \varepsilon g(\bar{\xi}, \varphi E_m) = 0. \end{aligned} \tag{2.8}$$

Therefore, by (2.5), (2.6), (2.7) and (2.8), (2.4) yields

$$(n-1)E_m(h) + \sum_{\alpha=1}^s \xi_\alpha(h) \eta^\alpha(E_m) = 0,$$

for any $m \in \{1, \dots, 2n+s\}$, from which it follows $(n-1)dh + \sum_{\alpha=1}^s \xi_\alpha(h) \eta^\alpha = 0$. Applying this 1-form to ξ_β , $\beta \in \{1, \dots, s\}$, one obtains $\xi_\beta(h) = 0$, for any $\beta \in \{1, \dots, s\}$. Hence, since $n \geq 2$ and $(n-1)X(h) = 0$ for any $X \in \text{Im } \varphi$, the claim follows. \square

3. INDEFINITE \mathcal{S} -SPACE FORMS AS η -EINSTEIN MANIFOLDS

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite \mathcal{S} -manifold and $p \in M$. For any non-lightlike unit vector $x \in \text{Im}(\varphi_p)$, the sectional curvature of (M, g) at p with respect to the plane $\pi = \text{span}\{x, \varphi x\}$ is called, by definition, the φ -sectional curvature of M at p , with respect to the φ -plane π . When it is independent of the choice of the φ -plane at any point, the manifold M is said to have *pointwise constant φ -sectional curvature*. An indefinite \mathcal{S} -manifold with pointwise constant φ -sectional curvature is said to be an *indefinite \mathcal{S} -space form* if the φ -sectional curvature does not depend on the point.

In [3] it is shown that an indefinite \mathcal{S} -manifold has pointwise constant φ -sectional curvature $c \in \mathfrak{F}(M)$ if, and only if, the Riemannian $(0, 4)$ -type curvature tensor field R of M satisfies the following identity

$$\begin{aligned} R(X, Y, Z, W) = & \frac{c+3\varepsilon}{4} \{g(\varphi X, \varphi Z)g(\varphi Y, \varphi W) - g(\varphi Y, \varphi Z)g(\varphi X, \varphi W)\} \\ & + \frac{c+\varepsilon}{4} \{\Phi(X, Z)\Phi(Y, W) - \Phi(Y, Z)\Phi(X, W) \\ & \quad + 2\Phi(X, Y)\Phi(Z, W)\} \\ & + \{\bar{\eta}(X)\bar{\eta}(Z)g(\varphi Y, \varphi W) - \bar{\eta}(Y)\bar{\eta}(Z)g(\varphi X, \varphi W) \\ & \quad + \bar{\eta}(Y)\bar{\eta}(W)g(\varphi X, \varphi Z) - \bar{\eta}(X)\bar{\eta}(W)g(\varphi Y, \varphi Z)\}, \end{aligned} \quad (3.1)$$

for any $X, Y, Z, W \in \Gamma(TM)$.

Many examples of \mathcal{S} -manifolds with indefinite metrics have been introduced and studied in different contexts. Namely, it is possible to endow \mathbb{R}^4 , \mathbb{R}^6 and $U(2)$ with non-trivial indefinite \mathcal{S} -structures. In particular, \mathbb{R}^4 and $U(2)$ turn out to be both Lorentzian \mathcal{S} -space forms, and it is easy to check that they are η -Einstein with $h = 0$ and $h = 4$, respectively (see [3] for more details about the non-compact examples, and [4] for the $U(2)$ case).

Now, we are going to show that any indefinite \mathcal{S} -manifold with pointwise constant φ -sectional curvature is η -Einstein.

Theorem 3.1. *Let $(M^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$, $n \geq 2$ and $s \geq 1$, be an indefinite \mathcal{S} -manifold with pointwise constant φ -sectional curvature $c \in \mathfrak{F}(M)$. Then M is η -Einstein.*

Proof. Let $(E_i, \varphi E_i, \xi_\alpha)$, $i \in \{1, \dots, n\}$ and $\alpha \in \{1, \dots, s\}$, be a local orthonormal φ -adapted frame. We have

$$\text{Ric}(X, Y) = \sum_{i=1}^n \varepsilon_i \{R(X, E_i, Y, E_i) + R(X, \varphi E_i, Y, \varphi E_i)\} + \sum_{\alpha=1}^s \varepsilon_\alpha R(X, \xi_\alpha, Y, \xi_\alpha).$$

for any $X, Y \in \Gamma(TM)$. Using (3.1) we have, for any $i \in \{1, \dots, n\}$ and any $\alpha \in \{1, \dots, s\}$:

$$\begin{aligned} R(X, E_i, Y, E_i) &= \frac{c+3\varepsilon}{4} \{g(\varphi X, \varphi Y)\varepsilon_i - g(\varphi E_i, \varphi Y)g(\varphi X, \varphi E_i)\} \\ &\quad + 3 \frac{c-\varepsilon}{4} \Phi(X, E_i)\Phi(Y, E_i) + \bar{\eta}(X)\bar{\eta}(Y)\varepsilon_i, \\ R(X, \varphi E_i, Y, \varphi E_i) &= \frac{c+3\varepsilon}{4} \{g(\varphi X, \varphi Y)\varepsilon_i - g(E_i, \varphi Y)g(\varphi X, E_i)\} \\ &\quad + 3 \frac{c-\varepsilon}{4} \Phi(X, \varphi E_i)\Phi(Y, \varphi E_i) + \bar{\eta}(X)\bar{\eta}(Y)\varepsilon_i, \\ R(X, \xi_\alpha, Y, \xi_\alpha) &= g(\varphi X, \varphi Y). \end{aligned}$$

Therefore, for any $X, Y \in \Gamma(TM)$, we get

$$\begin{aligned}
\text{Ric}(X, Y) &= 2n \frac{c+3\varepsilon}{4} g(\varphi X, \varphi Y) \\
&\quad - \frac{c+3\varepsilon}{4} \sum_{i=1}^n \varepsilon_i \{g(\varphi E_i, \varphi Y)g(\varphi X, \varphi E_i) + g(\varphi X, \varphi E_i)g(\varphi Y, \varphi E_i)\} \\
&\quad + 3 \frac{c-\varepsilon}{4} \sum_{i=1}^n \varepsilon_i \{g(\varphi X, E_i)g(\varphi Y, E_i) + g(\varphi X, \varphi E_i)g(\varphi Y, \varphi E_i)\} \\
&\quad + 2n\bar{\eta}(X)\bar{\eta}(Y) + \varepsilon g(\varphi X, \varphi Y) \\
&= \frac{c+3\varepsilon}{4} (2n-1)g(\varphi X, \varphi Y) + 3 \frac{c-\varepsilon}{4} g(\varphi X, \varphi Y) \\
&\quad + 2n\bar{\eta}(X)\bar{\eta}(Y) + \varepsilon g(\varphi X, \varphi Y) \\
&= \frac{1}{2} \{n(c+3\varepsilon) + c-\varepsilon\} g(\varphi X, \varphi Y) + 2n\bar{\eta}(X)\bar{\eta}(Y).
\end{aligned} \tag{3.2}$$

Then M turns out to be η -Einstein. \square

Clearly, for an indefinite \mathcal{S} -manifold with pointwise constant φ -sectional curvature $c \in \mathfrak{F}(M)$, (3.2) yields (2.3) with $h = \frac{1}{2} \{n(c+3\varepsilon) + c-\varepsilon\}$ and Theorem 2.6 implies the following consequence.

Theorem 3.2. *Let $(M^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$, $n \geq 2$ and $s \geq 1$, be an indefinite \mathcal{S} -manifold with pointwise constant φ -sectional curvature $c \in \mathfrak{F}(M)$. Then c is a constant function on M , i.e. M is an indefinite \mathcal{S} -space form.*

The above result extends the ones of [14] to the semi-Riemannian setting. To conclude we want to give the following remark that states a relation between the η -Einstein notion on an indefinite \mathcal{S} -manifold and the Kähler-Einstein one.

Remark 3.3. In [4] it is stated that an indefinite Kähler structure (J, g') on a manifold N can be lifted to an indefinite \mathcal{S} -structure $(\varphi, \xi_\alpha, \eta^\alpha, g)$ on the total space M of a principal toroidal bundle, whose projection $\pi : (M, \varphi, \xi_\alpha, \eta^\alpha, g) \rightarrow (N, J, g')$ turns out to be a semi-Riemannian submersion with totally geodesic fibres. Looking at [9, p. 15 and p. 145], in this context the Ricci formulas yield

$$\text{Ric}(X, Y) = \text{Ric}'(X', Y') \circ \pi - 2g(\bar{\xi}, \bar{\xi})g(\varphi X, \varphi Y);$$

where X, Y are basic vector fields π -related to X', Y' . When N is an Einstein manifold, by the above formula, it is clear that M is an η -Einstein manifold.

REFERENCES

- [1] D.E. Blair, Geometry of manifolds with structural group $U(n) \times O(s)$, J. Differential Geom. **4** (1970), 155–167
- [2] D.E. Blair, G. Ludden and K. Yano, Differential geometric structures on principal toroidal bundles, Trans. Amer. Math. Soc. **181** (1973), 175–184.

- [3] L. Brunetti and A.M. Pastore, Curvature of a class of indefinite globally framed f -manifolds, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **51(99)** (2008), no. 3, 183–204.
- [4] L. Brunetti and A. M. Pastore, Examples of indefinite globally framed f -structures on compact Lie groups, accepted on *Publ. Math. Debrecen*.
- [5] J.L. Cabrerizo, L.M. Fernandez and M. Fernandez, The curvature tensor fields on f -manifolds with complemented frames, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Sect. I a Mat.* **36** no. 2 (1990), 151–161.
- [6] K.L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Academic Publishers, Dordrecht, **364**, 1996.
- [7] K.L. Duggal, S. Ianuş and A.M. Pastore, Maps interchanging f -structures and their harmonicity, *Acta Appl. Math.* **67** (2001), 91–115.
- [8] K.L. Duggal and R. Sharma, *Symmetries of Spacetimes and Riemannian Manifolds*, Kluwer Academic Publishers, Dordrecht, **487**, 1999.
- [9] M. Falcitelli, S. Ianus and A.M. Pastore, *Riemannian submersions and related topics*. World Sci. Publishing, River Edge, NJ, 2004.
- [10] S.I. Goldberg and K. Yano, On normal globally framed f -manifolds, *Tôhoku Math. J.* **22** (1970), 362–370.
- [11] S.I. Goldberg and K. Yano, Globally framed f -manifolds, *Illinois J. Math.* **15** (1971), 456–474.
- [12] S. Ishihara, Normal structure f satisfying $f^3 + f = 0$, *Kôdai Math. Sem. Rep.* **18** (1966), 36–47.
- [13] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. I, II, Interscience Publishers, New York, 1963, 1969.
- [14] M. Kobayashi and S. Tsuchiya, Invariant submanifolds of an f -manifold with complemented frames, *Kôdai Math. Sem. Rep.* **24** (1972), 430–450.
- [15] H. Nakagawa, f -structures induced on submanifolds in spaces, almost Hermitian or Kaehlerian, *Kôdai Math. Sem. Rep.* **18** (1966), 161–183.
- [16] H. Nakagawa, On framed f -manifolds, *Kôdai Math. Sem. Rep.* **18** (1966), 293–306.
- [17] T. Takahashi, Sasakian manifold with pseudo-Riemannian metric, *Tôhoku Math. J.* **21** (1969), 271–290.
- [18] J. Vanzura, Almost r -contact structures, *Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat.* **26** (1972), 97–115.
- [19] K. Yano, On a structure defined by a tensor field f of type $(1, 1)$ satisfying $f^3 + f = 0$, *Tensor N.S.* **14** (1963), 99–109.

DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF BARI “ALDO MORO”
 VIA E. ORABONA, 4
 70125 – BARI
 ITALY
E-mail address: `brunetti@dm.uniba.it`